

Graphs with no equal length cycles

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see [1], p.247, Problem 11). Y. Shi [8] proved that

$$f(n) \geq n + [(\sqrt{8n - 23} + 1)/2]$$

for $n \geq 3$. E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] proved that

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

Chunhui Lai [7] proved that

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \geq \sqrt{2.4}.$$

and conjectured that

$$\lim_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} = \sqrt{2.4}.$$

We think it is difficult to prove this conjecture. It would be nice to prove the following conjecture:

Conjecture [6].

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

In 1988, Shi[8] proved that

For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n - 15} - 3)]$.

In 1998, Guantao Chen, Jenő Lehel, Michael S. Jacobson, and Warren E. Shreve [3] proved that

$$f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$$

In 2001, E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] improved this lower bound significantly:

$$f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}}).$$

and conjectured that

$$\lim_{n \rightarrow \infty} \frac{f_2(n) - n}{\sqrt{n}} = 1.$$

It is easy to see that this Conjecture implies the (difficult) upper bound in the Erdős Turán Theorem [4][5](see [2]).

References

- [1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976).
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